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Chiral decomposition in the non-commutative Landau problem

P-M. Zhang* and P. A. Horvathy†‡

Institute of Modern Physics,

Chinese Academy of Sciences

Lanzhou (China)

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Abstract

The decomposition of the non-commutative Landau (NCL) system into two uncoupled one-dimensional chiral components, advocated by Alvarez, Gomis, Kamimura and Plyushchay [1], is generalized to nonvanishing electric fields. This allows us to discuss the main properties of the NCL problem including its exotic Newton-Hooke symmetry and its relation to the Hall effect. The "phase transition" when the magnetic field crosses a critical value determined by the non-commutative parameter is studied in detail.

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† Permanent address : *Laboratoire de Mathématiques et de Physique Théorique*, Tours University (France).

*Electronic address: zhpm-at-impcas.ac.cn

‡Electronic address: horvathy-at-lmpt.univ-tours.fr

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1. INTRODUCTION

In two remarkable papers Alvarez, Gomis, Kamimura and Plyushchay [1] pointed out that two uncoupled 1d chiral oscillators yield, when combined, an interesting non-commutative system in the plane. The purely-magnetic "exotic" Landau problem (NCLP) [2, 3] can, in particular, be obtained for a suitable choice of the parameters. The inclusion of the electric field which, for non-commutative particles, also induces an anomalous velocity term (missed in Ref. [1]), is important, though, in the study of a Bloch electron [4], and for both the ordinary and the anomalous Hall effects [5, 6], for example.

In this paper we generalize the chiral oscillator — exotic Landau-problem correspondence to non-vanishing electric fields. The examples of a constant and of a harmonic electric field are worked out explicitly. We also re-derive, along the lines indicated by Ref. [1], the recently found "exotic" Newton – Hooke symmetry [7, 8].

The chiral framework is particularly useful for quantization which can be achieved, just like classically [7], using conserved quantities alone.

The system becomes singular when the magnetic field takes a certain critical value B_c; the physical meaning of the "phase transition" when this critical value is crossed is studied in detail.

It is worth mentioning that chiral oscillators were used before in the ordinary Landau problem and used to explain the non-commutativity of the guiding center coordinates [9, 10].

2. EXOTIC DYNAMICS

Let us first summarize some of the main features of "exotic" particles. Such a particle, moving in a planar electromagnetic field $\boldsymbol{E}, B$, [assumed static for simplicity] is described by the equations [2, 3],

$$m^*\dot{x}^i = p^i - me\theta\varepsilon^{ij}E^j, \qquad \dot{p}^i = eB\varepsilon^{ij}\dot{x}^j + eE^i, \tag{2.1}$$

where m, e and θ are the mass, charge and non-commutative parameter, respectively, and $m^* = m(1 - eB\theta)$ is the effective mass. Note also, in the first relation, the "anomalous velocity term", which is only absent for either in the purely magnetic- ($\boldsymbol{E} = 0$) or in the commutative ($\theta = 0$) case, and is indeed responsible for the Hall behavior [2, 3, 6].

It has been proved recently that when the charges, masses, and non-commutative parameters of a collection of exotic particles satisfy the generalized Kohn conditions $e_a/m_a = \text{const}\,.$ and $e_a\theta_a = \text{const}\,.$, then the system of interacting "exotic" electrons splits into internal and center-of-mass motions, with the latter behaving as a single exotic particle [7]. It is enough to study one particle therefore.

Eqns. (2.1) derive from the symplectic form and Hamiltonian,

$$\Omega = dp^i \wedge dx^i + \frac{\theta}{2}\varepsilon^{ij} dp^i \wedge dp^j + \frac{eB}{2}\varepsilon^{ij} dx^i \wedge dx^j, \qquad H = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{x}), \tag{2.2}$$

respectively, through the "exotic" Poisson brackets

$$\{x^i, x^j\} = \frac{\theta}{1 - eB\theta}\,\varepsilon^{ij}, \qquad \{x^i, p^j\} = \frac{\delta^{ij}}{1 - eB\theta}, \qquad \{p^i, p^j\} = \frac{eB}{1 - eB\theta}\,\varepsilon^{ij}. \tag{2.3}$$

When the magnetic field takes the critical value

$$B = B_c = \frac{1}{e\theta}\,, \tag{2.4}$$

the system becomes singular : the determinant of the symplectic matrix is det $(\Omega_{\alpha\beta}) = (m^*/m)^2 = 0$, and consistency requires the Hall law,

$$p^i = me\theta\varepsilon^{ij}E^j, \qquad \dot{x}^i = \varepsilon^{ij}\frac{E^j}{B}\,, \tag{2.5}$$

to be satisfied [2, 3]. This point will be further developed in Sect. 6.

3. SPLITTING INTO CHIRAL COMPONENTS

Turning to our promised chiral decomposition, we only consider two particular cases, since we are not yet able to tackle the problem for an arbitrary potential.

A. Constant force

Following Ref. [1], let us now introduce chiral coordinates $\boldsymbol{X}_\pm$ on 4d phase space,

$$p^i = eB\varepsilon^{ij}X_-^j, \qquad x^i = X_+^i + X_-^i\,. \tag{3.1}$$

In these terms,

$$\Omega \;=\; \Omega_+ + \Omega_- = \Big\{\frac{eB}{2}\left(\varepsilon^{ij} dX_+^i \wedge dX_+^j\right)\Big\} \;-\; \Big\{(1 - eB\theta)\frac{eB}{2}\left(\varepsilon^{ij} dX_-^i \wedge dX_-^j\right)\Big\}, \tag{3.2}$$

$$H \;=\; H_+ + H_- = \Big\{-e\boldsymbol{E}\cdot\boldsymbol{X}_+\Big\} \qquad + \;\Big\{\frac{(eB)^2}{2m}\boldsymbol{X}_-^2 - e\boldsymbol{E}\cdot\boldsymbol{X}_-\Big\}. \tag{3.3}$$

The system splits therefore into two uncoupled systems, both with 2d phase spaces, described by respective coordinates $\boldsymbol{X}_{\pm}$. Note the negative sign in Ω_{-}, justifying the expression "chiral".

Off the critical case, $eB\theta \neq 1$, (3.2) implies the fundamental Poisson Brackets

$$\{X_+^i, X_+^j\} = -\frac{1}{eB}\,\varepsilon^{ij}, \qquad \{X_-^i, X_+^j\} = 0, \qquad \{X_-^i, X_-^j\} = \frac{1}{eB(1-eB\theta)}\,\varepsilon^{ij}. \tag{3.4}$$

The chiral equations of motion,

$$\dot{X}_+^i = \varepsilon^{ij}\frac{E^j}{B}\,, \qquad m^*\dot{X}_-^i = eB\varepsilon^{ij}\,X_-^j - m\,\varepsilon^{ij}\frac{E^j}{B}\,, \tag{3.5}$$

are solved as

$$X_+^i(t) \;=\; \varepsilon^{ij}\frac{E^j}{B}\,t + X_+^i(0), \tag{3.6}$$

$$X_-^i(t) \;=\; \Big[R\left(-\omega^* t\right)\left(\boldsymbol{X}_-(0) - \boldsymbol{Y}\right)\Big]^i + Y^i, \;\; \boldsymbol{Y} = \frac{m\boldsymbol{E}}{eB^2} = \text{const}, \;\; \omega^* = \frac{eB}{m^*}. \tag{3.7}$$

$X_+(t)$, physically the guiding center, drifts therefore perpendicularly to the electric field according to the Hall law, while $\boldsymbol{X}_-$ rotates with angular velocity $(-\omega^*)$. Then a look at (3.1) allows us to conclude that, off the critical case, our exotic particle moves like an ordinary charged particle but with modified cyclotron frequency, $\omega^* = eB/m^*$, where $m^* = m(1-eB\theta)$ is the critical mass. Let us observe that the body of the trajectory does not depend on θ, only its speed.

In the critical case $B = B_c = (e\theta)^{-1}$ cf. (2.4), the second term in the symplectic form (3.2) is turned off, and the $\boldsymbol{X}_-$-degrees of freedom are lost. The second eqn. in (3.5) can nevertheless be satisfied provided the initial condition is $\boldsymbol{X}_-(0) = \boldsymbol{Y}$. Then $\boldsymbol{X}_-$ remains frozen into the fixed value $\boldsymbol{X}_-(t) = \boldsymbol{Y}$, leaving us with the $\boldsymbol{X}_+$-equation alone in (3.5), solved as in (3.6). Then from (3.1) we infer that for vanishing m^* the "whirling" is eliminated and the motion is along straight lines according to the Hall law,

$$x^i(t) = X_+^i(t) + Y^i = \varepsilon^{ij}\frac{E^j}{B}\,t + x^i(0), \tag{3.8}$$

materializing the guiding center motion. The latter follows parallel lines with the same velocity which can start from any point of the plane, see FIG. 4.

The behavior when the magnetic field sweeps from very weak to very strong crossing the critical value B_c is further studied in Section 6.

B. Harmonic trap

The splitting (3.1) works for the purely magnetic case, but fails when we consider a harmonic trap with spring constant $k \neq 0$. Generalizing (3.1) as

$$p^i = \epsilon^{ij}\big(\omega_+ X_+^j + \omega_- X_-^j\big), \qquad \boldsymbol{x} = \boldsymbol{X}_+ + \boldsymbol{X}_-\,, \tag{3.9}$$

with $\omega_\pm$ to be determined, we find

$$\begin{aligned}\Omega \;=\; & \underbrace{(eB - 2\omega_+ + \omega_+^2\theta)}_{\mu_+}\,\frac{1}{2}\varepsilon^{ij} dX_+^i \wedge dX_+^j \;+\; \underbrace{(eB - 2\omega_- + \omega_-^2\theta)}_{\mu_-}\,\frac{1}{2}\varepsilon^{ij} dX_-^i \wedge dX_-^j \\ & + \Big(eB + \omega_+\omega_-\theta - (\omega_+ + \omega_-)\Big)\,\varepsilon^{ij}\big(dX_+^i \wedge dX_-^j\big),\end{aligned} \tag{3.10}$$

which splits into $\Omega_+ + \Omega_-$ when the coefficient of the cross term vanishes,

$$eB + \omega_+\omega_-\theta - (\omega_+ + \omega_-) = 0. \tag{3.11}$$

For the Hamiltonian we find instead

$$H = \frac{\boldsymbol{p}^2}{2m} + k\frac{\boldsymbol{x}^2}{2} = \underbrace{\frac{1}{2m}\big(mk + \omega_+^2\big)\boldsymbol{X}_+^2}_{H_+} \;+\; \underbrace{\frac{1}{2m}\big(mk + \omega_-^2\big)\boldsymbol{X}_-^2}_{H_-} + \Big(k + \frac{\omega_+\omega_-}{m}\Big)\boldsymbol{X}_+ \cdot \boldsymbol{X}_-\,,$$

which splits as $H = H_+ + H_-$ when

$$\omega_+\omega_- + mk = 0. \tag{3.12}$$

Solving for $\omega_\pm$ allows us to deduce that choosing

$$\omega_\mp = \frac{eB - mk\theta \pm \sqrt{(eB - mk\theta)^2 + 4mk}}{2} \tag{3.13}$$

separates the exotic Landau problem with harmonic potential into two chiral oscillators.

In the purely magnetic case $k = 0$ the previous choice (3.2) is recovered. Our formulae are also consistent with those in the ordinary (commutative) Landau problem [9, 10].

Off the critical case, $eB\theta \neq 1$, the Poisson Brackets read

$$\{X_+^i, X_+^j\} = -\frac{1}{\mu_+}\,\varepsilon^{ij}, \qquad \{X_-^i, X_+^j\} = 0, \qquad \{X_-^i, X_-^j\} = -\frac{1}{\mu_-}\,\varepsilon^{ij} \tag{3.14}$$

with $\mu_\pm$ those coefficients in (3.10), yielding separated equations of motion,

$$m\mu_\pm \dot{X}_\pm^i = -(mk + \omega_\pm^2)\varepsilon^{ij} X_\pm^j. \tag{3.15}$$

For $\mu_\pm \neq 0$ the latter are solved, putting $X_\pm = X^1_\pm + iX^2_\pm$, as

$$X_\pm(t) = e^{i\alpha_\pm t}X_\pm(0), \quad \text{where} \quad \alpha_\pm = \frac{mk+\omega^2_\pm}{m\mu_\pm}\,. \tag{3.16}$$

(In the purely magnetic case $k=0$, $\alpha_+ = 0$ and $\alpha_- = -eB/m^* = -\omega^*$ as it should be, cf. [2, 3].) Then $\boldsymbol{x}(t) = \boldsymbol{X}_+(t) + \boldsymbol{X}_-(t)$ yields the motions, some of which are depicted on FIGs 1, 2 and 5, respectively. Here, and in all subsequent figures, we choose $m = e = \theta = 1$, and also $k = 1$ except in FIG. 2. The [red] dotted line indicates the guiding center.

Amusingly, all our trajectories are obtained by combining uniform rotation following an "epicycle", whose center is rotating uniformly on a "deferent" circle — as suggested by Ptolemy of Alexandria in the 1st century AD [but for planetary motion]. The form of the trajectory depends on the relative lengths [initial conditions] of $\boldsymbol{X}_\pm(0)$ and on the frequencies, determined by the relative strengths of the magnetic and oscillator fields, respectively.

For the critical value $B = B_c$ in (2.4),

$$\omega^c_+ = -mk\theta, \quad \omega^c_- = \frac{1}{\theta}, \qquad \mu^c_+ = \frac{1}{\theta}(1+mk\theta^2)^2, \quad \mu^c_- = 0. \tag{3.17}$$

The vanishing of μ_- implies that the $\boldsymbol{X}_-$-oscillator drops out again from the symplectic form and the system becomes singular [18], while μ_+ never vanishes, since both the mass and the spring constant are supposed to be positive, $m, k > 0$. Then for $m^* \to 0$ eqn. (3.15) can only be satisfied when $\boldsymbol{X}_-(t) = 0$ for all t : consistency requires that the $\boldsymbol{X}_-$-oscillator be switched off. We are left, hence, with the 1d $\boldsymbol{X}_+$-oscillator alone. The motions are therefore *uniform rotations along circles about the origin*,

$$x(t) = X_+(t) = e^{i\alpha^c t}X_+(0), \qquad \alpha^c = \alpha^c_+ = \frac{k\theta}{1+mk\theta^2}, \tag{3.18}$$

which are in fact the Hall motions, cf. [2, 3]. The Hall law requires indeed that the velocity be perpendicular to the (radial) force; the momentum is in turn frozen into its "Hall" value in (2.5).

This result is consistent with the vanishing of $\boldsymbol{X}_-(t)$ in (3.9). Let us also stress that α^c in (3.18) is the *continuous limit* of (3.16) when $B \to B_c$.

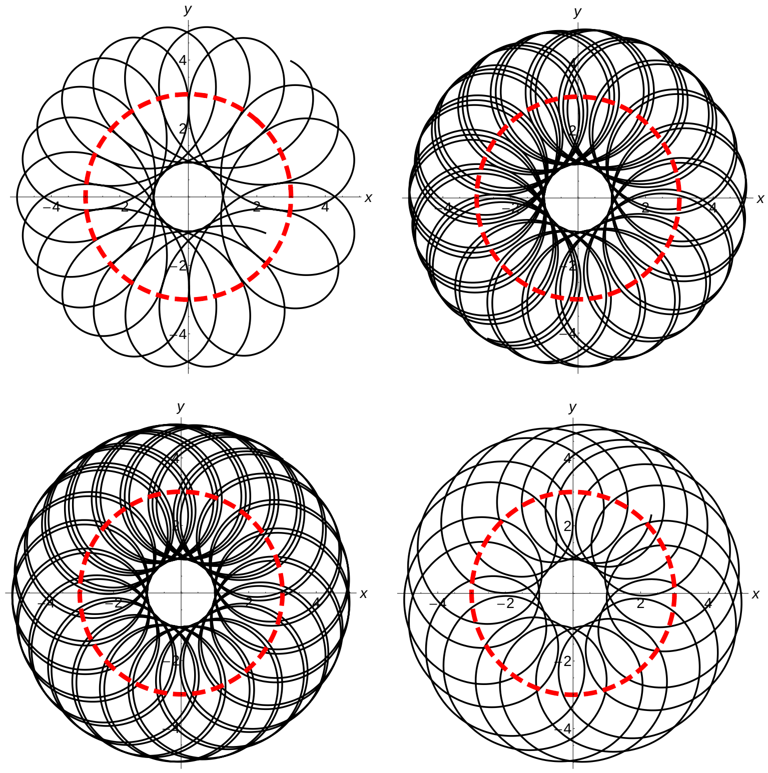

FIG. 1: *Trajectories in the exotic Landau problem with an oscillator with $eB\theta$ sweeping through the range of parameters $eB\theta = 0.5,\ 0.8,\ 1.1,\ 1.5$ and with initial conditions $X_+^1(0) = 2;\ X_+^2(0) = \sqrt{5};\ X_-^1(0) = 1;\ X_-^2(0) = \sqrt{3}$, respectively. The dotted circle in the middle indicates the trajectory of the guiding center.*

4. EXOTIC NEWTON-HOOKE SYMMETRY

It has been recognized recently [1, 7], that, off the critical case, $eB\theta \neq 1$, the NCLP admits the "exotic" two-fold central extension of the planar Newton-Hooke group as symmetry [8]. As hinted at in Ref. [1], the symmetry is readily recovered using the chiral decomposition.

A. Newton-Hooke symmetry in purely magnetic background

Let us first consider the purely magnetic case, $\boldsymbol{E} = 0$.

The $\boldsymbol{X}_+$-system is plainly invariant w.r.t. translations,

$$\boldsymbol{X}_+ \to \boldsymbol{X}_+ + \boldsymbol{c}, \tag{4.1}$$

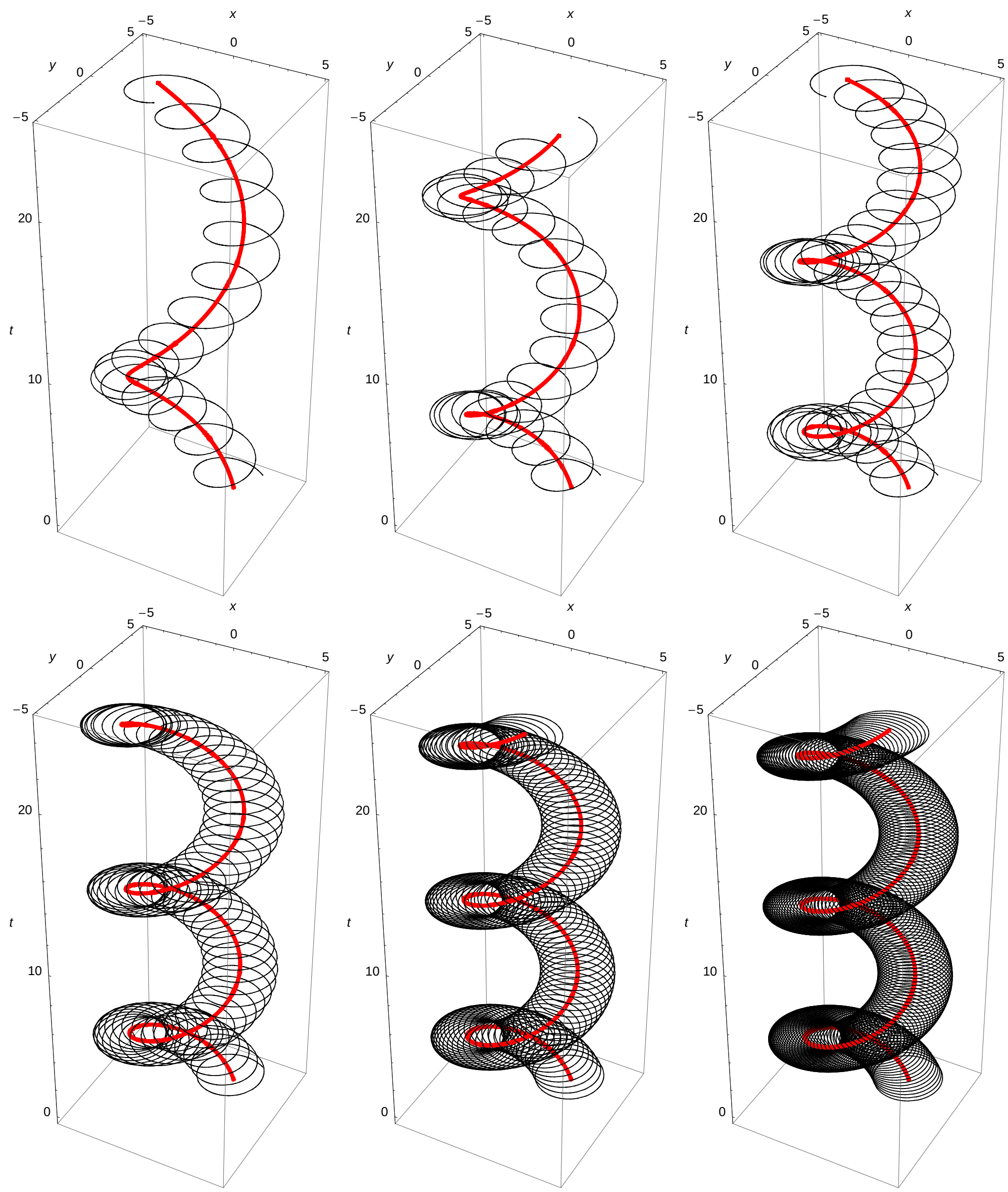

FIG. 2: *Trajectories unfolded into time with $eB\theta$ and the initial conditions kept constant but the spring constant k sweeping from very weak to very strong, $k = 0.5;\ 1;\ 2;\ 5;\ 10;\ 20$. With increasing k the "whirling" becomes stronger and stronger, whereas the guiding center motion changes slowly.*

for any constant vector $\boldsymbol{c}$, and the associated conserved quantity, calculated using

$$\Omega_+(\delta\boldsymbol{X}_+,\ \cdot\ ) = -d\mathcal{P}_+\,, \tag{4.2}$$

cf. [11], is readily found, as

$$\mathcal{P}^i_+ = -eB\varepsilon^{ij}X^j_+\,. \tag{4.3}$$

Its commutation relation, calculated using

$$\{\mathcal{P}_+^i, \mathcal{P}_+^j\} = -\Omega(\delta X_+^i, \delta X_+^j) \tag{4.4}$$

[11] is, moreover,

$$\{\mathcal{P}_+^i, \mathcal{P}_+^j\} = -eB\varepsilon^{ij}, \tag{4.5}$$

i.e., that of the *Heisenberg group* with central extension parameter $(-eB)$.

Extending the $\boldsymbol{X}_+$-only translations onto the whole phase space trivially, i.e., by $\boldsymbol{X}_- \to \boldsymbol{X}_-$ [so that $\mathcal{P}_-^i = 0$], we get ordinary translations $x^i \to x^i + c^i$, and our total conserved quantity, expressed in the original coordinates, reads

$$\boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{P}}_+ + \boldsymbol{\mathcal{P}}_-, \quad \text{where} \quad \mathcal{P}^i = p^i - eB\varepsilon^{ij}x^j, \tag{4.6}$$

as found before in [7]. Its commutation relations are still given in (4.5).

Similarly, the $\boldsymbol{X}_-$-system has Hamiltonian structure Ω_- and H_- which is that of a 1d harmonic oscillator, and for which

$$\boldsymbol{X}_- \to \boldsymbol{X}_- + R(-\omega^* t)\boldsymbol{a} \tag{4.7}$$

is a symmetry for any planar vector $\boldsymbol{a}$. The associated conserved quantity,

$$\boldsymbol{\mathcal{K}}_- = m(1 - eB\theta)^2\, R(\omega^* t)\dot{\boldsymbol{X}}_-, \tag{4.8}$$

satisfies

$$\{\mathcal{K}_-^i, \mathcal{K}_-^j\} = (1 - eB\theta)eB\,\varepsilon^{ij}, \tag{4.9}$$

i.e., the Heisenberg commutation relation but with "exotic" central parameter $(1-eB\theta)eB$. Extending the action (4.7) trivially to the $\boldsymbol{X}_+$-part provides us with "imported boost" symmetry $\boldsymbol{x} \to \boldsymbol{x} + R(-\omega^* t)\boldsymbol{a}$ [7, 8], and with [total] conserved quantity

$$\boldsymbol{\mathcal{K}} = (1 - eB\theta)\, R(\omega^* t)\, \boldsymbol{p}. \tag{4.10}$$

Planar rotations by angle φ, acting diagonally [1] i.e. as $\boldsymbol{X}_\pm \to R(\varphi)\boldsymbol{X}_\pm$, leave both dynamics invariant and provide us with conserved angular momenta, namely

$$\mathcal{J}_+ = \frac{eB}{2}\boldsymbol{X}_+^2, \qquad \mathcal{J}_- = -(1 - eB\theta)\frac{eB}{2}\boldsymbol{X}_-^2\,. \tag{4.11}$$

Expressed in terms of the original coordinates,

$$\mathcal{J} = \mathcal{J}_+ + \mathcal{J}_- = \boldsymbol{x} \times \boldsymbol{p} + \frac{eB}{2}\boldsymbol{x}^2 + \frac{\theta}{2}\,\boldsymbol{p}^2, \tag{4.12}$$

cf. [2, 7]. We mention, for completeness, that the Hamiltonian $H = \boldsymbol{p}^2/2$ is the conserved quantity associated with time translation symmetry $t \to t - \tau$ acting on the respective chiral oscillator phase spaces.

B. Newton-Hooke symmetry in magnetic + harmonic trap background

According to recent results, the exotic Newton-Hooke symmetry of the purely-magnetic problem can be extended to a constant electric field [7]. But in Sect. 3 A we have shown that the chiral decomposition remains valid even after switching on a constant electric field. It should hence be possible, in principle, to deduce its symmetries from those of the chiral components. For the $\boldsymbol{X}_+$ component this would be easy; in particular, $\boldsymbol{X}_+$-alone-translations are plainly symmetries : they merely shift the guiding center of the motion. The $\boldsymbol{X}_-$-alone-symmetries are, however, a bit more complicated, since the Hamiltonian H_- in (3.3) corresponds to a forced 1d oscillator, and the construction of [1] would therefore require extension. Considering the construction in [7] satisfactory, we do not pursue this issue here.

The Landau problem with a confining harmonic trap does, however, enter into the framework of Alvarez et al. [1], as the latter applies to any system which splits into two chiral oscillators. Spelling out their general formulas, we present here some details on the exotic Newton-Hook symmetry of the problem.

Firstly, the oscillator potential breaks the translation-invariance; both chiral oscillators admit instead independent "imported boost" symmetries, namely

$$\boldsymbol{\mathcal{K}}_\pm = m\mu_\pm^2\, R(-\alpha_\pm t)\dot{\boldsymbol{X}}_\pm, \tag{4.13}$$

with commutation relations

$$\{\mathcal{K}_\pm^i, \mathcal{K}_\pm^j\} = -\mu_\pm\, \epsilon^{ij}, \qquad \{\boldsymbol{\mathcal{K}}_+, \boldsymbol{\mathcal{K}}_-\} = 0. \tag{4.14}$$

The total angular momentum and energy,

$$\mathcal{J} \;=\; \mathcal{J}_+ + \mathcal{J}_-, \qquad \mathcal{J}_\pm = \frac{1}{2}\mu_\pm\, \boldsymbol{X}_\pm^2\,, \tag{4.15}$$

$$H \;=\; H_+ + H_-, \qquad H_\pm = \frac{mk + \omega_\pm^2}{2m}\, \boldsymbol{X}_\pm^2, \tag{4.16}$$

respectively, are associated to the diagonal action of rotations and time translations on the respective oscillator phase spaces. Note that $H_\pm = \alpha_\pm \mathcal{J}_\pm$.

5. QUANTIZATION

So far, all our investigations have been purely classical. It is not difficult to quantize our system, though, along the same lines as in the ordinary case [3, 8]. For simplicity, we only consider the purely magnetic $\boldsymbol{E} = 0$.

Classical quantities are promoted to operators [distinguished by "hats"]; Poisson brackets are replaced by $i/\hbar$-times commutators.

Assuming first that we are in the non-critical regime, $eB\theta \neq 1$, we observe that the quantum Hamiltonian is expressed in terms of the conserved quantity $\boldsymbol{\mathcal{K}}$ alone: putting

$$\hat{a} = \hat{\mathcal{K}}^1 + i\hat{\mathcal{K}}^2 \quad \text{and} \quad \hat{a}^\dagger = \hat{\mathcal{K}}^1 - i\hat{\mathcal{K}}^2, \tag{5.1}$$

the Hamiltonian reads indeed

$$\hat{H} = \hat{H}_- = \frac{\hat{a}^\dagger \hat{a}}{2m(1-eB\theta)^2} + \hbar\frac{\omega^*}{2}. \tag{5.2}$$

Note that $\hat{a}^\dagger$ and $\hat{a}$ are "$\boldsymbol{X}_-$-only operators"by (4.8). Owing to (4.9) we have,

$$[\hat{a}, \hat{a}^\dagger] = 2\hbar(1-eB\theta)m\omega. \tag{5.3}$$

$\hat{a}$ and $\hat{a}^\dagger$ are annihilation and creation operators acting on Fock space, $\hat{a}|0\rangle = 0$, so that $|n\rangle = (\hat{a}^\dagger)^n|0\rangle$, $n = 0,\, 1,\, \dots$ is an eigenvector of $\hat{a}^\dagger\hat{a}$ with eigenvalue $2\hbar(1-eB\theta)m\omega\, n$. It follows that the energy levels are

$$E_n = \hbar\omega^*\big(n + \frac{1}{2}\big) = \hbar\frac{eB}{m^*}\big(n+\frac{1}{2}\big), \quad n = 0, 1, \dots, \tag{5.4}$$

consistently with [3]. Note that the energy only depends on the $\boldsymbol{X}_-$-dynamics [19], its sign is that of $\mu_- = -eB(1-eB\theta)$ and flips therefore when the critical value is crossed.

In the same spirit, consider the oscillator representation of the other conserved quantity, *viz.* the magnetic momentum $\boldsymbol{\mathcal{P}}$,

$$\hat{b} = \hat{\mathcal{P}}^1 + i\hat{\mathcal{P}}^2, \quad \text{and} \quad \hat{b}^\dagger = \hat{\mathcal{P}}^1 - i\hat{\mathcal{P}}^2. \tag{5.5}$$

These are, by (4.3), "$\boldsymbol{X}_+$-only operators" which satisfy

$$[\hat{b}, \hat{b}^\dagger] = 2i\hbar m\omega. \tag{5.6}$$

$\hat{b}$ and $\hat{b}^\dagger$ are therefore again annihilation and creation operators, so that $|p\rangle = (\hat{b}^\dagger)^p|0\rangle$ is an eigenvector of $\hat{b}^\dagger\hat{b}$ with eigenvalues $2\hbar eB\,p$ with $p = 0, 1, 2, \ldots$. Our new oscillator-operators do not intervene into the energy, but they do enter into the *total angular momentum operator*,

$$\hat{J} = \frac{1}{2eB}\left(\hat{b}^\dagger\hat{b} - \frac{1}{1-eB\theta}\,\hat{a}^\dagger\hat{a}\right), \tag{5.7}$$

which has, therefore, eigenvalues [1]

$$J_{p,n} = \hbar\left(\mathrm{sg}(\mu_+)p - \mathrm{sg}(\mu_-)n + \frac{1}{2}(\mathrm{sg}(\mu_+) - \mathrm{sg}(\mu_-)\right), \qquad p, n = 0, 1, 2, \ldots \tag{5.8}$$

Thus, the spectrum of the angular momentum is unbounded in the "subcritical regime" $eB\theta < 1$ and becomes bounded from one side when we cross over into the "supercritical regime" $eB\theta > 1$ cf. [1, 3].

Our quantum Hilbert space is thus composed of wave functions

$$\psi = |p, n\rangle = |p\rangle\,|n\rangle. \tag{5.9}$$

The quantization in the critical case $B = B_c$ is postponed to the next Section.

Skipping details, we would like to mention that the Landau problem with confining oscillator symmetry could be quantized along the same line, namely by replacing the magnetic translational $\boldsymbol{\mathcal{P}}$-symmetry by the "imported boost symmetry", $\boldsymbol{\mathcal{K}}_+$.

6. STUDY OF THE "PHASE TRANSITION" $m^* \to 0$

A. Classical aspects

It is amusing to study the "phase transition" $m^* \to 0$ in some detail. Let us start with the classical mechanics. A look at our plots reveals that the trajectories seem, somewhat surprisingly, to keep their body shapes as $m^* \to 0$ — although we know that most of them become inconsistent. How can this come about ? A simple explanation is as follows: write the velocity-momentum relation [the first equation in (2.1), or its X_--counterpart in the chiral decomposition, eqn. (3.5) or in (3.15)] in the form

$$\mu_-\,dX^i_- = \Big(\ \ldots\ \Big)dt. \tag{6.1}$$

Then, as $\mu_- \to 0$, we have two alternatives:

1. either $\big(\dots \big) = 0$, which eliminates all initial conditions *except those which are consistent with the Hall law*;

2. or ... the "motion" should be *instantaneous*, $dt = 0$!

An intuitive understanding of the transition from the "subcritical" to "supercritical" regime, and of the appearance of our strange "instantaneous motions" in particular, is that, being proportional to $(\mu_-)^{-1}$, the frequency of the $\boldsymbol{X}_-$-oscillator goes to infinity as $B \to B_c$. This implies that the body of the trajectory is unchanged but it is described more and more rapidly. When $\mu_- = 0$ we have the singular case, and only the Hall motions keep on advancing in time at a regular [namely the Hall] velocity, whereas all other motions are "flattened" to $t = \text{const}$. Changing the sign of μ_- amounts, furthermore, to changing the orientation of the trajectories : our $\boldsymbol{X}_-$-oscillator flips over its chirality. All this is confirmed by the plots in FIG.s 3–4–5 below. FIG. 5 provides, in the oscillator case, particular insight : when the critical value is approached, the "non-Hall" trajectories speed up since the frequencies, inversely proportional to μ_-, diverge. But the frequency of the central line representing the guiding center, which performs Hall motion, remains finite, namely as given in (3.18).

Another interesting aspect is that when the critical value B_c is crossed, all commutation relations in (2.3), change sign.

B. Quantization in the critical phase

Turning to quantization in the critical case, we observe that, when $m^* \to 0$, only those quantum states can survive which are killed by $\hat{a}^\dagger \hat{a}$, due to the $(1 - eB\theta)^{-2}$ factor in the Hamiltonian (5.2). Such states belong therefore to a subspace of the original Hilbert space, — namely that of the *lowest Landau level*, characterized by $n = 0$ in (5.4). Although the ground-state energy E_0 diverges as $m^* \to 0$, the "good" [meaning "Hall"] states themselves *do* survive the transition: they belong to the Fock space of $\hat{b}$ and $\hat{b}^\dagger$ alone, $|p\rangle = (\hat{b}^\dagger)^p |0\rangle$, where $\hat{b}|0\rangle = 0$.

The reduced Hilbert space can also be represented by the "Bargmann-Fock" wave functions of the form

$$\psi(z) = f(z) e^{-|z|^2/4\theta}, \tag{6.2}$$

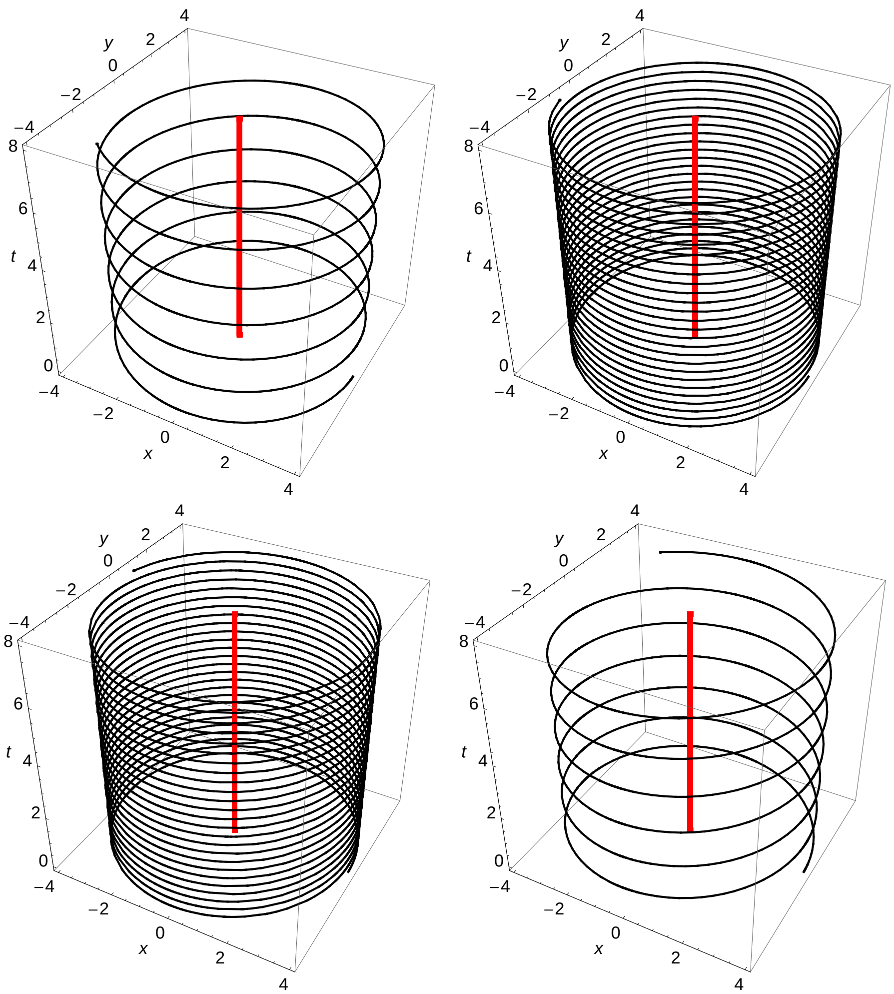

FIG. 3: *Trajectories in the purely-magnetic problem for* $eB\theta = 0.8,\ 0.95,\ 1.05,\ 1.20$. *When* $B\theta < 1$ *is increased starting from the weak-field regime, the trajectory "speeds up" and becomes hence " denser" until it reaches the critical regime. For* $eB\theta = 1$, *all trajectories become "instantaneous" — with the exception of the "Hall ones", which are now simple points. After crossing the critical value, the chirality is reversed and the curves change orientation. This also explains why the angular momentum becomes bounded from one side after crossing the critical value.*

with $f(z)$ analytic in $z = (X_+^1 + iX_+^2)$ [2, 3]. Physicists have long appreciated this point [5, 13, 14].

It is the Hamiltonian operator $\hat{H}$ which changes when $B \to B_c$, namely by renormalizing $\hat{H}$ by subtracting the ground-state term $\hbar\omega^*/2$. It becomes, hence, *identically zero* in the purely magnetic case, or the potential alone, V, otherwise. Moreover, $\boldsymbol{X}_-$ is frozen and the dynamical degree of freedom represented by it is lost, so that our reduced "Hamiltonian" [i.e. the potential] is a function of $\boldsymbol{X}_+$ alone. The latter has, however *non-commuting*

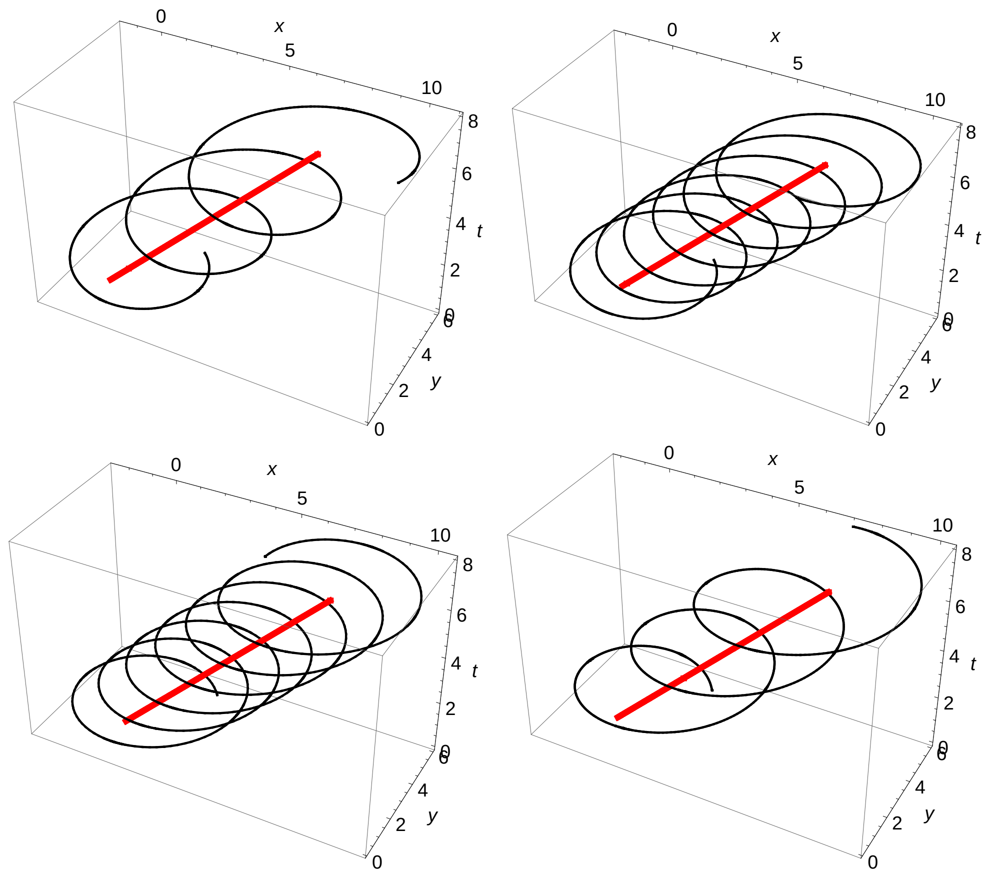

FIG. 4: *Trajectories in the magnetic plus constant electric field problem unfolded into time, with the electric field oriented into the second direction and $eB\theta$ sweeping through the values* 0.6, 0.8, 1.20, 1.4*. The behavior is similar to the one in the purely magnetic problem, except for the Hall drift of the guiding center.*

coordinates.

This is the "Peierls substitution" [12, 16], which has become a standard tool in the theory of the QHE [5].

The reduced angular momentum in the critical case is that of the $\boldsymbol{X}_+$-part alone,

$$\mathcal{J}^c = \mathcal{J}_+ = \frac{eB_c}{2}\,\widehat{\boldsymbol{X}_+^2} = \frac{\theta}{2}\,\hat{b}^\dagger\hat{b}, \tag{6.3}$$

consistently with turning off the second term in (4.11); it is clearly bounded from one side.

At last, the reduced Hamiltonian is $\hat{V}$ written as a function of the non-commuting $\widehat{\boldsymbol{X}}_+$ alone. In the purely magnetic case, it is thus *zero*: no motion is left over.

For a constant electric field, one should consider plane waves and a continuous spectrum.

For the harmonic trap, it is that of the $\boldsymbol{X}_+$ oscillator alone,

$$H^c = H^c_+ = (1 + mk\theta^2)\,\frac{k}{2}\,\widehat{\boldsymbol{X}^2_+} = \theta^2(1 + mk\theta^2)\,\frac{k}{2}\,\hat{b}^\dagger\hat{b}\,, \tag{6.4}$$

which should be compared with the naive expression $(k/2)\boldsymbol{x}^2$. The reduced energy is hence proportional to the angular momentum,

$$H^c = k\,\theta(1 + mk\theta^2)\,\mathcal{J}^c. \tag{6.5}$$

It is worth to mention that for a general (non linear or quadratic) potential V the very quantization procedure is ambiguous [3].

7. CONCLUSION

Proceeding backwards, we generalized to non-vanishing electric fields the remarkable correspondence between the NCLP and chiral oscillators, advocated before by Alvarez et al. [1]. Our clue has been replacing the "natural" velocity-momentum relation posited in [1],

$$\dot{\boldsymbol{x}} = \boldsymbol{p}/m, \tag{7.1}$$

by including the anomalous velocity term as it is usual in the non-commutative context [see the first condition in (2.1)] and, more importantly, it is required by the physical applications. It is the removal of this usual but unjustified rule which resolves the problem — just like for anyons [15].

Details have been worked out for both a constant electric field and and a harmonic map, providing us with a quick and elegant derivation of the principal properties of the NCL problem, cf. [2, 3].

Particular interest has been devoted to the phase transition which takes place when the magnetic field crosses the critical value $B_c = (e\theta)^{-1}$, when the effective mass, m^*, changes sign. It is intriguing to investigate what happens in the "Hall regime" $B = B_c$ i.e. $m^* = 0$, when only those motions are consistent which satisfy the Hall law. This is understood by realizing that, when $m^* \to 0$, the rotation around the guiding center becomes more and more rapid — with the exception, precisely, of those motions which materialize that of the guiding center. The initial conditions fall hence into two categories: those which correspond to the Hall law survive unscathed, whereas the others speed up more and more. Such trajectories become hence more and more dense, becoming "instantaneous" as $m^* \to 0$.

Do these "instantaneous motions" have any physical sense ? It is hard to say. A footnote in Ref. [13] hints, however, at that they *may* contribute to the path integral : according to an unpublished statement of Klauder, there is finite propagation in infinitesimally short time.

The quantum mechanical description can, just like in classical mechanics [7], be derived from the symmetry alone. This is analogous to the similar derivation of the H-atom spectrum form the group theory, using the O(4) dynamical symmetry.

The transition in the critical case is analogous to the one described in "Chern-Simons mechanics" [16], obtained by turning off the kinetic term in the ordinary Landau problem.

Let us mention, in conclusion, that the chiral decomposition has recently been extended to the Hill equations of celestial mechanics [17].

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[18] This can also be seen from noting that det $[\Omega_{\alpha\beta}] = \mu_+^2\, \mu_-^2$.

[19] The degeneracy of the energy levels is plainly lifted when an electric field is added.

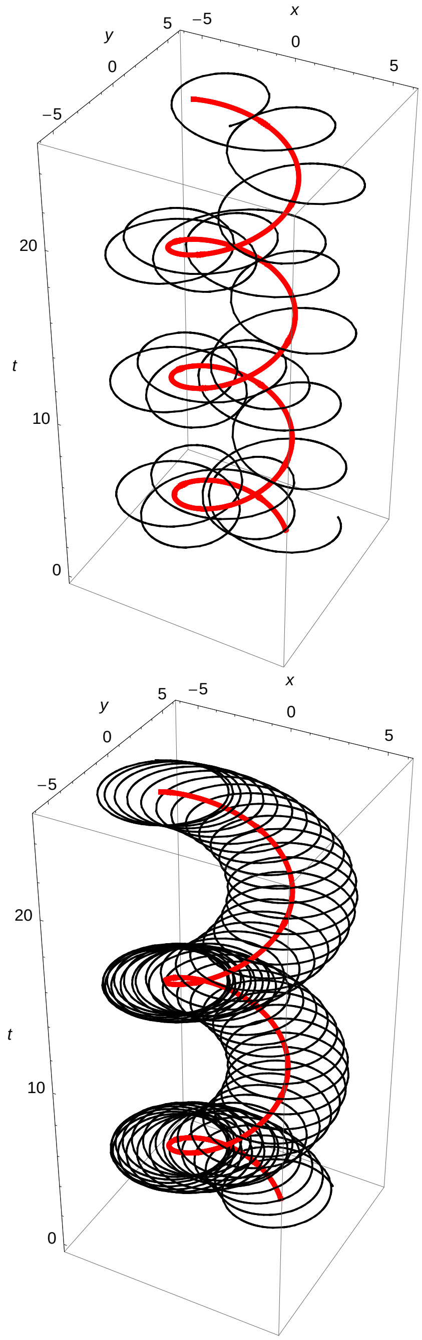

FIG. 5: *Trajectories in the magnetic- plus harmonic trap problem unfolded into time, for* $k = 2$ *and* $eB\theta = 0.5,\ 0.8,\ 1.20,\ 1.5$. *The asymmetry between the sub- and supercritical regimes is caused by the harmonic background. When the critical value is approached, "non-Hall" trajectories speed up but the frequency of the central line indicating the guiding center remains finite.*